

COLLAPSING OF NEGATIVE KÄHLER-EINSTEIN METRICS

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ABSTRACT. In this paper, we study the collapsing behaviour of negative Kähler-Einstein metrics along degenerations of canonical polarized manifolds. We prove that for a toroidal degeneration of canonical polarized manifolds with the total space \mathbb{Q} -factorial, the Kähler-Einstein metrics on fibers collapse to a lower dimensional complete Riemannian manifold in the pointed Gromov-Hausdorff sense by suitably choosing the base points. Furthermore, the most collapsed limit is a real affine Kähler manifold.

1. INTRODUCTION

Let X be a complex projective n -manifold. We call X a canonical polarized manifold if the canonical bundle \mathcal{K}_X of X is ample, and call X a Calabi-Yau manifold if \mathcal{K}_X is trivial. The Calabi conjecture of the existence of Kähler-Einstein metrics was solved by Aubin and Yau in the case of canonical polarized manifolds (cf. [1, 40]), and by Yau for Calabi-Yau manifolds (cf. [40]). More precisely, on a canonical polarized manifold X , there exists a unique Kähler-Einstein metric ω with $\omega \in 2\pi c_1(\mathcal{K}_X)$ and negative Ricci curvature, i.e.

$$\mathrm{Ric}(\omega) = -\omega,$$

by [1, 40]. On a Calabi-Yau manifold, there are Ricci-flat Kähler-Einstein metrics by [40]. The goal of this paper is to study the collapsing behaviour of families of negative Kähler-Einstein metrics along degenerations in algebro-geometric sense.

A degeneration of projective n -manifolds $\pi : \mathcal{X} \rightarrow \Delta$ is a flat morphism from a normal Gorenstein variety \mathcal{X} of dimension $n+1$ to a disc $\Delta \subset \mathbb{C}$ such that $X_t = \pi^{-1}(t)$, $t \in \Delta^* = \Delta \setminus \{0\}$, is smooth except the central fiber $X_0 = \pi^{-1}(0)$. We denote $X_0 = \bigcup_{i=1}^l X_{0,i}$ and $X_{0,I} = \bigcap_{i \in I} X_{0,i}$, where $X_{0,i}$, $i = 1, \dots, l$, is a irreducible component, and $I \subset \{1, \dots, l\}$. If the relative canonical bundle $\mathcal{K}_{\mathcal{X}/\Delta} = \mathcal{K}_{\mathcal{X}} \otimes \mathcal{K}_{\Delta}^{-1}$ is relatively ample, then for any smooth fiber X_t , the canonical bundle $\mathcal{K}_{X_t} \cong \mathcal{K}_{\mathcal{X}/\Delta}|_{X_t}$ is ample, and thus X_t is a canonical polarized manifold. We call such degeneration $\pi : \mathcal{X} \rightarrow \Delta$ a canonical polarized degeneration.

The author is supported in part by grant NSFC-11271015.

In [35], Strominger, Yau and Zaslow proposed a conjecture, so called SYZ conjecture, for constructing mirror Calabi-Yau manifolds via dual special lagrangian fibration. Later, a new version of the SYZ conjecture was proposed by Kontsevich, Soibelman, Gross and Wilson (cf. [17, 25, 26]) by using the collapsing of Ricci-flat Kähler-Einstein metrics. Let $\mathcal{X} \rightarrow \Delta$ be a degeneration of Calabi-Yau n -manifolds, i.e. the relative canonical bundle $\mathcal{K}_{\mathcal{X}/\Delta}$ is trivial, and $0 \in \Delta$ be a large complex limit point (cf. [14]). The collapsing version of SYZ conjecture asserts that there are Ricci-flat Kähler-Einstein metrics ω_t on X_t for $t \in \Delta^*$ such that $(X_t, \text{diam}_{\omega_t}^{-2}(X_t)\omega_t)$ converges to a compact metric space (B, d_B) in the Gromov-Hausdorff sense, when $t \rightarrow 0$. Furthermore, the smooth locus B_0 of B is open dense, and is of real dimension n , and admits a real affine structure. The metric d_B is induced by a Monge-Ampère metric g_B on B_0 , i.e. under affine coordinates x_1, \dots, x_n , there is a potential function ϕ such that

$$g_B = \sum_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j, \quad \text{and} \quad \det \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = 1.$$

Clearly it is true for Abelian varieties. This conjecture was verified by Gross and Wilson for fibred K3 surfaces with only type I_1 singular fibers in [17], and was studied for higher dimensional HyperKähler manifolds in [15, 16]. In [16], Gross-Wilson's result was extended to all elliptically fibred K3 surfaces.

Inspired by this collapsing version of SYZ conjecture, we study the limits of negative Kähler-Einstein metrics on canonical polarized manifolds degenerating to some singular varieties.

Let $\pi : \mathcal{X} \rightarrow \Delta$ be a canonical polarized degeneration such that X_0 has only simple normal crossing singularities, i.e. X_0 is reduced, locally given by $z_1 \cdots z_k = 0$ under local coordinates z_1, \dots, z_n on \mathcal{X} , and any $X_{0,I}$ is smooth. Let $\omega_t \in 2\pi c_1(\mathcal{K}_{X_t})$, $t \in \Delta^*$, be the unique Kähler-Einstein metric on X_t . The convergence of ω_t was studied by various authors (cf. [36, 19, 29, 30, 32]). In [36], it is proved that ω_t converges smoothly to a complete Kähler-Einstein ω_0 with negative Ricci curvature on the regular

locus $X_{0,reg} = \bigcup_{i=1}^l X_{0,i,reg}$ in the Cheeger-Gromov sense, if an additional condition that any three of the components $X_{0,i}$ have empty intersection is satisfied. More precisely, for any smooth family of embeddings $F_t : X_{0,reg} \rightarrow X_t$, we have that

$$F_t^* \omega_t \rightarrow \omega_0, \quad \text{when } t \rightarrow 0,$$

in the locally C^∞ -sense on $X_{0,reg}$, where ω_0 is the complete Kähler-Einstein metric on $X_{0,reg}$ previously obtained in [37, 22, 5]. In [29, 19], the additional assumption is removed, and furthermore, the result is generalized to the case of toroidal degenerations in [30]. These theorems describe the non-collapsing part of the limit of (X_t, ω_t) .

Since the volume of ω_0 is finite, there must be some collapsing part when (X_t, ω_t) approaches to the limit, i.e. there are points $p_t \in X_t$ such that the

volumes of metric 1-balls satisfy

$$\text{Vol}_{\omega_t}(B_{\omega_t}(p_t, 1)) \rightarrow 0, \quad \text{when } t \rightarrow 0.$$

Now by Gromov's precompactness theorem (cf. [12]), a sequence of (X_t, ω_t, p_t) converges to a pointed complete metric space (W, d_W, p_∞) of Hausdorff dimension less than $2n$ in the pointed Gromov-Hausdorff sense, i.e. for any $R > 0$, the metric R -ball $(B_{\omega_t}(p_t, R), \omega_t)$ converges to the metric R -ball $(B_{d_W}(p_\infty, R), d_W)$ in the Gromov-Hausdorff sense (cf. [8]).

The following theorem is a special case of the main theorem (Theorem 2.4) of the present paper, where a more general hypothesis is assumed.

Theorem 1.1. *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a canonical polarized degeneration such that X_0 has only simple normal crossing singularities, and $\omega_t \in 2\pi c_1(\mathcal{K}_{X_t})$ be the unique Kähler-Einstein metric on X_t , $t \in \Delta^*$. For any $X_{0,I}$ and any point $p_0 \in X_{0,I} \setminus \bigcup_{i \notin I} X_{0,i}$, there are points $p_t \in X_t$ such that $p_t \rightarrow p_0$ in \mathcal{X} when $t \rightarrow 0$, and by passing to a sequence, (X_t, ω_t, p_t) converges to a complete Riemannian manifold (W, g_W, p_∞) with $\dim_{\mathbb{R}} W = 2n + 1 - \sharp I$ in the pointed Gromov-Hausdorff sense. Furthermore, if $\dim_{\mathbb{C}} X_{0,I} = 0$, then (W, g_W) is isometric to (B, g_B) by suitably choosing p_t , where B is the interior of the standard simplex in \mathbb{R}^n , and there is a smooth potential function ϕ on B such that $\phi|_{\partial B} = +\infty$,*

$$g_B = \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j, \quad \text{and} \quad \det \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \kappa e^{2\phi},$$

for a constant $\kappa > 0$.

Actually (X_t, ω_t) collapses smoothly in a certain sense, which is stronger than the Gromov-Hausdorff topology (See Theorem 2.4 for details).

This theorem shows a similar collapsing behaviour to the SYZ conjecture for Calabi-Yau manifolds, i.e. under certain assumptions, the limit metric space W is an affine Kähler manifold of real dimension n , and the potential function satisfies a real Monge-Ampère equation. However, unlike the Calabi-Yau case, we always have the non-collapsing part of the limit, and we do not rescale the metric to obtain the collapsing limit. Note that for algebraic curves of higher genus, the rescaled limit exists, and is a compact metric graph by [28]. However, we do not expect that still holds in the higher dimensional case.

In the original SYZ conjecture (cf. [35]), the existence of special lagrangian submanifolds is expected when Calabi-Yau manifolds are near the large complex limit. As an application, we will construct some generalized special lagrangian submanifolds on canonical polarized manifolds (See Section 2.3 for details).

The understanding of the limit behaviour of negative Kähler-Einstein metrics is also required for other program. The moduli space \mathcal{M} of canonical polarized manifolds with a fixed Hilbert polynomial was proven to be

a quasi-projective manifold by Viehweg in [39], and the recent progress on the moduli space of stable varieties (cf. [23]) gives a natural algebro geometric compactification $\overline{\mathcal{M}}$ of \mathcal{M} . Meanwhile, the existence of singular Kähler-Einstein metrics on stable varieties was obtained in [2]. A natural question is to understand such compactification from the differential geometric viewpoint (cf. [2, 34]), for example in the Gromov-Hausdorff sense or the Weil-Petersson geometry sense. However unlike the case of Calabi-Yau manifolds (cf. [41, 38]), we would not have the coincidence of the Gromov-Hausdorff non-collapsing convergence and the finite Weil-Petersson distance. In Theorem 1.1, (X_t, ω_t) diverges in the Gromov-Hausdorff sense, but the Weil-Petersson metric on Δ^* is not complete, i.e. $\{0\}$ has finite Weil-Petersson distance to the interior by [36, 29, 30].

This paper is organized as the followings. In Section 2, we introduce the preliminary materiel and state the main theorems (Theorem 2.4 and Theorem 2.6) of this paper. In Section 1.1, we construct some semi-flat Kähler-Einstein metrics from those affine Kähler metrics obtained by Cheng and Yau previously. In Section 1.2 and Section 1.3, the main theorems (Theorem 2.4 and Theorem 2.6) are given. Theorem 2.4 study the metric collapsing along toroidal degenerations, and Theorem 2.6 shows the existence of generalized special lagrangian submanifolds. Section 3 is devoted to prove Theorem 2.4. Firstly, we construct the approximation background metrics in Section 3.1, then we do some local calculations and prove Theorem 2.4 in Section 3.2. The last section proves Theorem 2.6.

Acknowledgements: The author would like to thank Prof. Shiu-Yuen Cheng for answering a question.

2. MAIN THEOREMS

In this paper, we always denote $N \cong \mathbb{Z}^{n+1}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$, $M = \text{hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

2.1. Semi-flat Kähler-Einstein metric. In this section, we recall a theorem due to Cheng and Yau for the existence of affine Kähler metrics, which induce some semi-flat Kähler-Einstein metrics that appear in the main theorem.

Let σ be a rational strongly convex polyhedral cone in $N_{\mathbb{R}}$, and $\check{\sigma} \subset M_{\mathbb{R}}$ be the dual cone. If $u_{\sigma} \in M \cap \check{\sigma}$ satisfies $\langle u_{\sigma}, v \rangle = 1$ for the primitive lattice vector $v \in \tau \cap N$ of any 1-dimensional face τ of σ , then we define

$$\Lambda_{\mathbb{R}} = \{v \in N_{\mathbb{R}} | \langle v, u_{\sigma} \rangle = 1\}, \quad B_{\sigma} = \Lambda_{\mathbb{R}} \cap \text{Int}(\sigma), \quad \text{and} \quad \Lambda = N \cap \Lambda_{\mathbb{R}}$$

where $\text{Int}(\sigma)$ denotes the interior of σ . The closure \overline{B}_{σ} of B_{σ} is a rational convex polytope in $\Lambda_{\mathbb{R}}$.

Let \mathcal{Y}_{σ} be the affine toric variety associated to σ , i.e. $\mathcal{Y}_{\sigma} = \text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$, and $t = \mathcal{Z}^{u_{\sigma}} : \mathcal{Y}_{\sigma} \rightarrow \mathbb{C}$. We have a family of varieties $Y_{\sigma, t} = \text{div}(\mathcal{Z}^{u_{\sigma}} - t)$

degenerating to the toric boundary Y_0 , i.e. $Y_0 = \bigcup_{i=1}^d D_i$ where D_i is a primitive toric Weil divisor.

If $e_0, \dots, e_n \in N$ is a basis, we denote x_0, \dots, x_n the respecting coordinates on $N_{\mathbb{R}}$, and denote $z_j = \mathcal{Z}^{e_j^*}$, $j = 0, \dots, n$. If $u_\sigma = \sum_{j=0}^n m_j e_j^*$, then $Y_{\sigma,t}$ is given by $z_0^{m_0} \dots z_n^{m_n} = t$, and $\Lambda_{\mathbb{R}}$ is given by $m_0 x_0 + \dots + m_n x_n = 1$. Without loss of generality, we assume that x_1, \dots, x_n are coordinates on $\Lambda_{\mathbb{R}}$, i.e. $m_0 \neq 0$, which give an integral affine structure on B_σ .

For any $t \in \Delta^*$, the logarithmic map is

$$\text{Log}_t : T_N \rightarrow N_{\mathbb{R}}, \quad \text{by } z_j \mapsto x_j = \frac{\log |z_j|}{\log |t|}, \quad j = 0, \dots, n.$$

It is clear that $\text{Log}_t(Y_{\sigma,t}) = \Lambda_{\mathbb{R}}$. We denote

$$\mathcal{U} = \{p \in \mathcal{Y}_\sigma \mid |\mathcal{Z}^{u_k}(p)| < 1, k = 1, \dots, d'\},$$

which is an open subset of \mathcal{Y}_σ , where $u_k \in M \cap \check{\sigma}$ such that $\sigma = \{v \in N_{\mathbb{R}} \mid \langle v, u_k \rangle \geq 0, k = 1, \dots, d'\}$. We have $\text{Log}_t(\mathcal{U}) = \text{Int}(\sigma)$, and moreover, $\text{Log}_t(Y_{\sigma,t} \cap \mathcal{U}) = B_\sigma$.

We define coordinates $\theta_1, \dots, \theta_n$ on $\Lambda_{\mathbb{R}}$ by $\theta_j = dx_j$, $j = 1, \dots, n$, under the identification of the tangent bundle $TB_\sigma \cong B_\sigma \times \Lambda_{\mathbb{R}}$. Then there is a natural complex structure on $B_\sigma \times \sqrt{-1}\Lambda_{\mathbb{R}}$ given by complex coordinates $w_j = x_j + \sqrt{-1}\theta_j$, $j = 1, \dots, n$, which induces a complex structure on $Y_{t,m_0}(B_\sigma) = B_\sigma \times \sqrt{-1}(\Lambda_{\mathbb{R}} / \frac{2\pi m_0 \Lambda}{\log |t|})$ for any $t \in \Delta^*$. We define a finite covering map $q_\sigma : Y_{t,m_0}(B_\sigma) \rightarrow Y_{\sigma,t} \cap \mathcal{U}$ by setting $z_j = \exp((\log |t|)w_j)$, $j = 1, \dots, n$, and

$$z_0 = \exp\left(\frac{1}{m_0} \log |t| + \sqrt{-1} \frac{\arg(t)}{m_0} - \sum_{j=1}^n \frac{m_j}{m_0} (\log |t|) w_j\right).$$

Furthermore, $f_t = \text{Log}_t|_{Y_{\sigma,t} \cap \mathcal{U}} : Y_{\sigma,t} \cap \mathcal{U} \rightarrow B_\sigma$ is a fibration such that $f_t \circ q_\sigma$ is the projection from $Y_{t,m_0}(B_\sigma)$ to B_σ .

Now we recall a theorem for the existence of affine Kähler metrics in [7].

Theorem 2.1 (Theorem 4.4 in [7]). *For any constant $\kappa > 0$, there is a smooth convex solution ϕ of the real Monge-Ampère equation*

$$(2.1) \quad \det \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \kappa e^{2\phi}, \quad \phi|_{\partial \overline{B}_\sigma} = +\infty,$$

and

$$g_{B_\sigma} = \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j$$

is a complete affine Kähler metric on B_σ .

Note that the constant κ is chosen to be 1 in [7], and however, we can obtain the general case by rescaling the coordinates. By pulling back ϕ , we

regard ϕ as a function on $B_\sigma \times \sqrt{-1}\Lambda_{\mathbb{R}}$, i.e. $\phi(w_1, \dots, w_n) = \phi(x_1, \dots, x_n)$, which defines a complete Kähler metric

$$(2.2) \quad \omega^{sf} = 2\sqrt{-1}\partial\bar{\partial}\phi = \frac{\sqrt{-1}}{2} \sum_{ij=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} dw_i \wedge d\bar{w}_j$$

on $B_\sigma \times \sqrt{-1}\Lambda_{\mathbb{R}}$. By (2.1), ϕ satisfies the complex Monge-Ampère equation $\det\left(\frac{\partial^2 \phi}{\partial w_i \partial \bar{w}_j}\right) = 4^n \kappa e^{2\phi}$ on $B_\sigma \times \sqrt{-1}\Lambda_{\mathbb{R}}$, and hence ω^{sf} is a Kähler-Einstein metric with Ricci curvature -1 , i.e.

$$\text{Ric}(\omega^{sf}) = -\sqrt{-1}\partial\bar{\partial} \log \det\left(\frac{\partial^2 \phi}{\partial w_i \partial \bar{w}_j}\right) = -\omega^{sf}.$$

Now Proposition 5.5 in [6] implies that ϕ is the unique solution of (2.1) (See also [18]).

Since both ϕ and ω^{sf} are invariant under the translation $w_j \mapsto w_j + \sqrt{-1}\lambda$ for any $\lambda \in \mathbb{R}^1$, ω^{sf} descends to a complete Kähler-Einstein metric on $Y_{t,m_0}(B_\sigma)$ first, for any $t \in \Delta^*$, and further to a complete Kähler-Einstein metric on $Y_{\sigma,t} \cap \mathcal{U}$ denoted by ω_t^{sf} . Note that the corresponding Riemannian metric of ω^{sf} is

$$g^{sf} = \sum_{ij=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} (dx_i dx_j + d\theta_i d\theta_j).$$

The first consequence is that the restriction of ω_t^{sf} on any fiber $f_t^{-1}(x)$, $x \in B_\sigma$, is flat, so called a semi-flat Kähler-Einstein metric. The second one is that the diameter of the fiber

$$\text{diam}_{\omega_t^{sf}}(f_t^{-1}(x)) \sim -(\log |t|)^{-1} \rightarrow 0,$$

and by suitably choosing a family of base points $p_t \in Y_{\sigma,t}$, $(Y_{\sigma,t} \cap \mathcal{U}, \omega_t^{sf}, p_t)$ converges to $(B_\sigma, g_{B_\sigma}, p_\infty)$ in the pointed Gromov-Hausdorff sense, when $t \rightarrow 0$. We say that $(Y_{\sigma,t} \cap \mathcal{U}, \omega_t^{sf})$ collapses to (B_σ, g_{B_σ}) .

In summary, we have the following proposition.

Proposition 2.2. *For any $t \in \Delta^*$, there is a unique complete Kähler-Einstein metric ω_t^{sf} on $Y_{\sigma,t} \cap \mathcal{U}$ such that the Ricci curvature is -1 , i.e.*

$$\text{Ric}(\omega_t^{sf}) = -\omega_t^{sf},$$

and ω_t^{sf} is semi-flat respecting to the torus fibration $f_t : Y_{\sigma,t} \cap \mathcal{U} \rightarrow B_\sigma$. Furthermore, $(Y_{\sigma,t} \cap \mathcal{U}, \omega_t^{sf}, p_t)$ converges to $(B_\sigma, g_{B_\sigma}, p_\infty)$ in the pointed Gromov-Hausdorff sense by choosing a family of base points $p_t \in Y_{\sigma,t}$, when $t \rightarrow 0$.

The logarithm Log_t is used to convert classical algebraic varieties to tropical varieties (cf. [27]), and it is believed that the collapsing of Kähler-Einstein metrics can do the same in certain circumstances (cf. [13, 9]). This is true in our case as a direct corollary of the previous arguments.

Let $\mathbf{p} \in \mathbb{C}[\check{\sigma} \cap M](t)$, i.e. $\mathbf{p} = \sum_{u \in A} b_u t^{v(u)} \mathcal{Z}^u$ for a finite set $A \subset \check{\sigma} \cap M$, $b_u \in \mathbb{C}^*$, and $v : A \rightarrow \mathbb{Z}$, and $V_{t,\mathbf{p}} \subset Y_{\sigma,t}$ be the variety defined by $\mathbf{p}|_{Y_{\sigma,t}} = 0$. The image $\mathcal{A}_t = \text{Log}_t(V_{t,\mathbf{p}}) \subset \Lambda_{\mathbb{R}}$ is called an amoeba, and it is proven in [27] that \mathcal{A}_t converges to a polyhedron complex \mathcal{A}_{∞} in the Hausdorff topology, when $t \in \mathbb{R}$ and $t \rightarrow 0$. Here \mathcal{A}_{∞} is called a non-Archimedean amoeba, and is the set of non-smooth points of the function

$$\mathbf{p}_{\infty}(x) = \min_{u \in A} \{v(u) + \langle x, u \rangle\}$$

on $\Lambda_{\mathbb{R}}$. In tropical geometry, \mathcal{A}_{∞} is the tropical hypersurface defined by \mathbf{p} (cf. [27]). We have the following corollary by the collapsing of ω_t^{sf} to $g_{B_{\sigma}}$.

Corollary 2.3. *When $t \in \mathbb{R}$ and $t \rightarrow 0$,*

$$V_{t,\mathbf{p}} \cap \mathcal{U} \rightarrow \mathcal{A}_{\infty} \cap B_{\sigma}$$

under the pointed Gromov-Hausdorff convergence of $(Y_{\sigma,t} \cap \mathcal{U}, \omega_t^{sf})$ to $(B_{\sigma}, g_{B_{\sigma}})$.

2.2. Toroidal degeneration. A degeneration $\pi : \mathcal{X} \rightarrow \Delta$ is called simple toroidal, if for any point $x \in \mathcal{X}$, there is an open neighborhood U satisfying that

- i) U is isomorphic to an open subset of an affine toric variety \mathcal{Y}_{σ} , denoted still by U .
- ii) The restriction of π on U is given by a regular function $\mathcal{Z}^{u_{\sigma}}$, where $u_{\sigma} \in M \cap \check{\sigma}$ satisfies $\langle u_{\sigma}, v \rangle = 1$ for the primitive lattice vector $v \in \tau \cap N$ of any 1-dimensional face τ of σ . Hence if D_1, \dots, D_d are primitive toric Weil divisors of \mathcal{Y}_{σ} , then we have that $X_0 \cap U = \sum_{j=1}^d D_j \cap U$, and X_0 is reduced.
- iii) Any non-empty $X_{0,I}$ is connected and normal, which implies that any $X_{0,I}$ does not intersect with itself.

Since the canonical divisor $\mathcal{K}_{\mathcal{Y}_{\sigma}} = -\sum_{j=1}^d D_j$ (cf. [10]), we have that $\mathcal{K}_{\mathcal{X}}|_U = -\text{div}(\mathcal{Z}^{u_{\sigma}})$, and thus $\mathcal{K}_{\mathcal{X}}$ is Cartier, i.e. \mathcal{X} is Gorenstein. Degenerations with only simple normal crossing singularities are special cases of simple toroidal degenerations.

In Chapter II of [21], a compact polyhedral complex \mathcal{B} with integral structure, called the dual intersection complex, is associated to $\pi : \mathcal{X} \rightarrow \Delta$ such that cells of \mathcal{B} are in one-to-one correspondence to those non-empty $X_{0,I}$. More precisely, for any $X_{0,I} \neq \emptyset$, there is a unique polyhedral cell $\overline{B}_I \in \mathcal{B}$ such that $\dim_{\mathbb{R}} \overline{B}_I = n - \dim_{\mathbb{C}} X_{0,I}$, and $\overline{B}_{I'}$ is a face of \overline{B}_I if and only if $X_{0,I'} \supset X_{0,I}$. The cell $\overline{B}_I \in \mathcal{B}$ associated to $X_{0,I}$ is constructed as the following. Let $p \in X_{0,I} \setminus \bigcup_{j \notin I} X_{0,j}$, and $U \subset \mathcal{X} \setminus \bigcup_{j \notin I} X_{0,j}$ be a neighborhood of p isomorphic to an open subset of an affine toric variety \mathcal{Y}_{σ} . If σ is the

corresponding rational convex cone in $N_{\mathbb{R}}$, then

$$\overline{B}_I = \{v \in \sigma \mid \langle v, u_{\sigma} \rangle = 1\}.$$

We denote B_I the interior of \overline{B}_I .

Now we state the main theorem of the present paper.

Theorem 2.4. *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a simple toroidal canonical polarized degeneration of projective n -manifolds, and ω_t be the unique Kähler-Einstein metric in $2\pi c_1(K_{X_t})$, $t \in \Delta^*$. If \mathcal{X} is \mathbb{Q} -factorial, then the followings hold.*

- i) *For any $X_{0,I}$ with $\sharp I > 1$, and any point $p_0 \in X_{0,I} \setminus \bigcup_{i \notin I} X_{0,i}$, there are points $p_t \in X_t$ such that $p_t \rightarrow p_0$ in \mathcal{X} when $t \rightarrow 0$, and by passing to a sequence, (X_t, ω_t, p_t) converges to a complete Riemannian manifold (W, g_W, p_{∞}) with $\dim_{\mathbb{R}} W = \dim_{\mathbb{R}} \overline{B}_I + 2 \dim_{\mathbb{C}} X_{0,I}$ in the pointed Gromov-Hausdorff sense.*
- ii) *If $\dim_{\mathbb{C}} X_{0,I} = 0$, then (W, g_W) is isometric to (B_I, g_{B_I}) by suitably choosing p_t , where g_{B_I} is the complete affine Kähler metric obtained in Theorem 2.1. Furthermore, if $\omega_{t,I}^{sf}$ is the semi-flat Kähler-Einstein metric constructed from g_{B_I} in Proposition 2.2 on a neighborhood of $U \cap X_t$, where U is a neighborhood of $X_{0,I}$ isomorphic an open subset of a toric variety, then*

$$\|\omega_t - \omega_{t,I}^{sf}\|_{C_{loc}^{\nu}(X_t \cap U, \omega_{t,I}^{sf})} \rightarrow 0,$$

for any $\nu > 0$, when $t \rightarrow 0$, i.e. the collapsing is in the C^{∞} -sense, and the convergence do not need to pass any sequence.

This theorem describes the collapsed limits of ω_t , while the previous results of [36, 19, 29, 30] describe the non-collapsed limits, i.e. they still have complex dimension n .

The notion of toroidal degeneration is an algebro-geometric analogue of F -structure introduced in [3]. An F -structure \mathcal{F} on a smooth manifold X consists an open covering $\{U_{\alpha}\}$ such that for each U_{α} , there is an effective $T^{n_{\alpha}}$ -action on a finite cover of U_{α} , and on any overlap $U_{\alpha} \cap U_{\beta}$, these two torus actions $T^{n_{\alpha}}$ and $T^{n_{\beta}}$ are compatible in a certain sense (See [9] for the details). For a toroidal degeneration $\pi : \mathcal{X} \rightarrow \Delta$, a small neighborhood U of a $X_{0,I}$ with $\sharp I > 1$ is isomorphic to an open subset of a toric variety, and $X_t \cap U$ is given by a monomial. Thus there is a natural local $T^{n_{\alpha}}$ -action on $X_t \cap U$. We conjecture that there is an F -structure \mathcal{F} on $X_t \cap \mathfrak{U}$, where \mathfrak{U} is a small neighborhood of $\bigcup_{\sharp I > 1} X_{0,I}$ in \mathcal{X} , and more importantly, this \mathcal{F} is

Hamiltonian, i.e. there is a symplectic form ϖ_t on X_t such that any local torus action of \mathcal{F} is Hamiltonian.

Theorem 2.4 and Proposition 3.4 in Section 3.2 show that the Kähler-Einstein metric ω_t approximates some local semi-flat Kähler-Einstein metrics $\omega_{t,I}^{sf}$ on small open subsets of X_t , and $\omega_{t,I}^{sf}$ collapses smoothly to lower dimensional spaces along local torus fibrations. Moreover, we would see that

the curvature of ω_t is bounded independent of t in Section 3.1. Hence there is an F -structure \mathcal{F}' on some region of X_t by [4], and we again conjecture that \mathcal{F}' can be made to coincide with the above \mathcal{F} . Hamiltonian F -structures would be studied in a separate paper.

We remark that Theorem 2.4 should hold for more general settings, for example, toroidal degenerations without the assumption of \mathcal{X} being \mathbb{Q} -factorial as in [31], or the log pair case, i.e. $\mathcal{K}_{\mathcal{X}/\Delta} + D$ is ample for a Cartier divisor D , as in [36, 19]. For avoiding too many technique difficulties, we leave those generalizations for future studies. In a recent paper [2], the existence of singular Kähler-Einstein metrics is obtained for stable varieties, i.e. varieties with semi-log canonical singularities and ample canonical divisor. It is also expected that the convergence theorems of [36, 19, 29, 30, 32] can be generalized to degenerations with central fiber X_0 stable varieties (cf. [2, 34]), which is related to the question of differential geometric understanding of the moduli space for stable varieties.

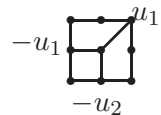
We finish this section by showing an example that Theorem 2.4 and Theorem 1.1 can apply.

Example 2.5. Firstly, we recall the standard Mumford degeneration of toric varieties. Let $M' \cong \mathbb{Z}^n$ such that $M \cong M' \times \mathbb{Z}$, and $\mathcal{P} \subset M'_{\mathbb{R}} = M' \otimes_{\mathbb{Z}} \mathbb{R}$ be a lattice polytope. If $\psi : \mathcal{P} \rightarrow \mathbb{R}$ is a piecewise linear convex function respecting to a lattice polyhedral decomposition \mathfrak{P} of \mathcal{P} with integral slopes, we define a lattice polyhedron

$$\tilde{\mathcal{P}} = \{(v, r) \in M_{\mathbb{R}} \cong M'_{\mathbb{R}} \times \mathbb{R} \mid \psi(v) \leq r\},$$

which determines a toric variety $X_{\tilde{\mathcal{P}}}$ with a regular function $\pi = \mathcal{Z}^{(0,1)} : X_{\tilde{\mathcal{P}}} \rightarrow \mathbb{C}$. For any $t \in \mathbb{C} \setminus \{0\}$, $X_t = \pi^{-1}(t)$ is isomorphic to the toric variety $X_{\mathcal{P}}$ associated to \mathcal{P} , and $X_0 = \pi^{-1}(0) = \bigcup_{\tau \in \mathfrak{P}_{\max}} X_{\tau}$, where \mathfrak{P}_{\max} denotes the

set of n -dimensional polytopes of \mathfrak{P} , and X_{τ} is the toric variety associated to $\tau \in \mathfrak{P}_{\max}$. By choosing \mathcal{P} and ψ properly, we can assume that X_0 has only simple normal crossing singularities, and X_t is smooth for any $t \neq 0$. For instance, we take \mathcal{P} , \mathfrak{P} and ψ as the following:



$$\psi(-u_1) = 0, \psi(-u_2) = 0, \psi(u_1 + u_2) = 1.$$

Now we follow the argument in the proof of Lemma 1.4 in [24]. Let H be a sufficiently general very ample divisor on $X_{\tilde{\mathcal{P}}}$ such that $\mathcal{K}_{X_{\tilde{\mathcal{P}}}} \otimes \mathcal{O}(H)$ is ample, and $H + X_t$ has simple normal crossing singularities for any $|t| < \varepsilon \ll 1$. If $\mathbf{c} : \tilde{X}_{\tilde{\mathcal{P}}} \rightarrow X_{\tilde{\mathcal{P}}}$ is the double ramified cover along $2H$, then the Hurwitz formula shows that $\mathcal{K}_{\tilde{X}_{\tilde{\mathcal{P}}}} \cong \mathbf{c}^*(\mathcal{K}_{X_{\tilde{\mathcal{P}}}} \otimes \mathcal{O}(H))$, and hence, $\mathcal{K}_{\tilde{X}_{\tilde{\mathcal{P}}}}$ is ample. Note that $\tilde{X}_0 = \mathbf{c}^{-1}(X_0)$ still has only simple normal crossing singularities, and for any t with $0 < |t| \ll 1$, $\tilde{X}_t = \mathbf{c}^{-1}(X_t)$ is smooth. We obtain a canonical degeneration $\tilde{\pi} : \mathcal{X} \rightarrow \Delta \subset \mathbb{C}$ satisfying the hypotheses in Theorem 2.4 and Theorem 1.1 by letting $\tilde{\pi} = \pi \circ \mathbf{c}$ and $\mathcal{X} = \tilde{\pi}^{-1}(\Delta)$.

2.3. Special lagrangian submanifold. The original SYZ conjecture asserts the existence of special lagrangian submanifolds when Calabi-Yau manifolds are near the large complex limit (cf. [35]). There are some attempts to generalize the SYZ conjecture to the case of canonical polarized manifolds (cf. [20]), which include analog notions for special lagrangian submanifold. We also like to study a generalization of special lagrangian submanifold.

If X is a canonical polarized projective n -manifold, then by definition, the canonical bundle \mathcal{K}_X is ample. Let Ω be a holomorphic n -form, and D be the effective divisor defined by Ω , i.e. $D = \text{div}(\Omega)$. The restriction of Ω on $X \setminus D$ is no-where vanishing, and thus $\mathcal{K}_{X \setminus D}$ is trivial, i.e. $X \setminus D$ is a quasi-projective Calabi-Yau manifold. A submanifold L of $X \setminus D$ is called a generalized special lagrangian submanifold respecting to Ω and a Kähler metric ω , if $\dim_{\mathbb{R}} L = n$,

$$\omega|_L \equiv 0, \quad \text{and} \quad \text{Im}(\Omega)|_L \equiv 0.$$

This notion of generalized special lagrangian submanifold is standard in the case of non-Ricci flat metric (cf. [14, 31]). The real part $\text{Re}(\Omega)$ is not a calibration respecting to the Kähler metric ω , but to a non-Kähler Hermitian metric $\rho\omega$ by Section 10.5 in [14], where $\rho > 0$ is a function defined by $\rho^n \omega^n = \frac{n!}{2^n} (-1)^{\frac{n^2}{2}} \Omega \wedge \bar{\Omega}$.

As an application of Theorem 2.4, we have the following theorem.

Theorem 2.6. *Let $\pi : \mathcal{X} \rightarrow \Delta$ and ω_t be the same as in Theorem 2.4. Assume that there is a zero dimensional $X_{0,I}$. If Ω_t is a section of $\mathcal{K}_{\mathcal{X}/\Delta}$ such that $D = \text{div}(\Omega_t)$ does not intersect with $X_{0,I}$, then there is a generalized special lagrangian torus $L_t \subset (X_t \setminus D_t)$ respecting to ω_t and $e^{\sqrt{-1}\vartheta_t} \Omega_t|_{X_t}$ for any $0 < |t| \ll 1$ and a phase $\vartheta_t \in \mathbb{R}$, where $D_t = D \cap X_t$.*

3. PROOF OF THEOREM 2.4

3.1. Background metric. In this section, we use the construction in [29] to obtain some approximation background Kähler metrics, which are uniformly equivalent to Kähler-Einstein metrics.

Let $\pi : \mathcal{X} \rightarrow \Delta$ be a simple toroidal canonical polarized degeneration of projective n -manifolds such that \mathcal{X} is \mathbb{Q} -factorial. Since $\mathcal{K}_{\mathcal{X}/\Delta}$ is relative ample, there is an embedding $\Phi : \mathcal{X} \hookrightarrow \mathbb{CP}^{N_m} \times \Delta$ for two integers $m > 0$ and $N_m > 0$ such that $\mathcal{K}_{\mathcal{X}/\Delta}^m \cong \Phi^* \mathcal{O}_{\mathbb{CP}^{N_m}}(1)$. There are sections $\Psi_0, \dots, \Psi_{N_m}$ of

$\mathcal{K}_{\mathcal{X}/\Delta}^m$ such that, by abusing notions, $h_{FS} = (\sum_{k=0}^{N_m} |\Psi_k|^2)^{-\frac{1}{m}}$ is the Hermitian metric whose curvature is the Fubini-Study metric, i.e.

$$(3.1) \quad \omega^o = \Phi^* \left(\frac{1}{m} \omega_{FS} + \sqrt{-1} dt \wedge d\bar{t} \right) = \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{k=0}^{N_m} |\Psi_k|^2 \right)^{\frac{1}{m}}.$$

By regarding volume forms as Hermitian metrics of the anti-canonical bundle, we obtain a volume form $V = (\sum_{k=0}^{N_m} |\Psi_k|^2)^{\frac{1}{m}}$ on \mathcal{X} . For any $t \in \Delta^*$, $V_t = V \otimes (dt \wedge d\bar{t})^{-1}$ is a smooth volume form on X_t , and let

$$(3.2) \quad \omega_t^o = \omega^o|_{X_t} = \sqrt{-1} \partial \bar{\partial} \log V_t.$$

Since \mathcal{X} is \mathbb{Q} -factorial, there is a $\mu \in \mathbb{N}$ such that all of $\mu X_{0,i}$, $i = 1, \dots, l$, are Cartier divisors. Let $\|\cdot\|_i$ be a smooth Hermitian metric of $\mathcal{O}(\mu X_{0,i})$ on \mathcal{X} , and s_i be a defining section of $\mu X_{0,i}$, i.e. $\text{div}(s_i) = \mu X_{0,i}$. Here the Hermitian metric $\|\cdot\|_i$ being smooth means that $\|\cdot\|_i$ is locally given by the restriction of a smooth positive function ϱ on the ambient space \mathbb{C}^ν for a local embedding of an open subset U of \mathcal{X} into \mathbb{C}^ν , and a trivialization of $\mathcal{O}(\mu X_{0,i})$ on U . In this case, $\text{Ric}(\|\cdot\|_i)$ is the restriction of the smooth form $-\sqrt{-1} \partial \bar{\partial} \log \varrho$ on \mathbb{C}^ν .

We assume that $s_1 \cdots s_l = t^\mu$ by choosing the parameter $t \in \Delta$ appropriately. Let

$$(3.3) \quad \alpha_i = \frac{1}{\mu} \log \|s_i\|_i^2, \quad \chi_t = (\log |t|^2)^2 \prod_{i=1}^l \alpha_i^{-2},$$

and

$$(3.4) \quad \begin{aligned} \tilde{\omega}_t &= \sqrt{-1} \partial \bar{\partial} \log \chi_t V_t \\ &= \omega_t^o + \sqrt{-1} \partial \bar{\partial} \log \chi_t \\ &= \omega_t^o + 2 \sum_{i=1}^l \left(\frac{\text{Ric}(\|\cdot\|_i)}{\alpha_i} + \sqrt{-1} \frac{\partial \alpha_i \wedge \bar{\partial} \alpha_i}{\alpha_i^2} \right) |_{X_t} \end{aligned}$$

on X_t for $t \neq 0$. We can assume that $\|s_i\|_i \leq \varepsilon \ll 1$ such that

$$\frac{1}{2} \omega^o \leq \omega^o + \sum_{i=1}^l \frac{2}{\alpha_i} \text{Ric}(\|\cdot\|_i) \leq 2\omega^o$$

on $\mathcal{X} \setminus X_0$ by multiplying certain constants if necessary. We denote $X_{0,I}^o = X_{0,I} \setminus \bigcup_{i \notin I} X_{0,i}$, and define a complete Kähler metric

$$(3.5) \quad \tilde{\omega}_{0,I} = \omega^o|_{X_{0,I}^o} + 2 \sum_{i \notin I} \left(\frac{\text{Ric}(\|\cdot\|_i)}{\alpha_i} + \sqrt{-1} \frac{\partial \alpha_i \wedge \bar{\partial} \alpha_i}{\alpha_i^2} \right) |_{X_{0,I}^o}$$

on $X_{0,I}^o$.

The Kähler metric $\tilde{\omega}_t$ is the background metric we need. Note that our assumption of \mathcal{X} is stronger than the one in [29], and however is weaker than that in [30]. Nevertheless, the arguments in Section 3 of [29] and Section 4 of [30] show that the curvature of $\tilde{\omega}_t$ and the Ricci potential $\log(\frac{V_t}{\tilde{\omega}_t^n})$ are bounded independent of t , which can also be obtained by the calculation in Section 3.2. Thus we have the C^0 and C^2 estimates for the potential function

of the Kähler-Einstein metric by the standard estimates for Monge-Ampère equations (cf. [1, 40]).

Proposition 3.1. *Let φ_t be the unique solution of Monge-Ampère equation*

$$(3.6) \quad (\tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{\varphi_t} \chi_t V_t,$$

and $\omega_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$ be the Kähler-Einstein metric on X_t . Then

$$|\varphi_t| \leq C_1, \quad \text{and} \quad C_2^{-1}\tilde{\omega}_t \leq \omega_t \leq C_2\tilde{\omega}_t,$$

for constants $C_1 > 0$ and $C_2 > 0$ independent of t .

Once Proposition 3.1 is obtained, [36, 19, 29, 30] prove the convergence of ω_t to a complete Kähler-Einstein metric ω_0 on the regular locus $X_{0,reg}$ in the Cheeger-Gromov sense, i.e. for any smooth family of embeddings $F_t : X_{0,reg} \rightarrow X_t$, $F_t^* \omega_t$ converges to ω_0 in the locally C^∞ -sense when $t \rightarrow 0$. When $\sharp I = 1$, $\tilde{\omega}_{0,I}$ is uniformly equivalent to the Kähler-Einstein metric ω_0 on $X_{0,I}^o \subset X_{0,reg}$.

3.2. Proof of Theorem 2.4. Now we study the local collapsing behaviour of Kähler-Einstein metrics ω_t .

For a point $p \in X_{0,I}$, let $U \subset \mathcal{X}$ be a neighborhood of p isomorphic to an open subset of a toric variety \mathcal{Y}_σ , denoted still by U , such that $U \cap X_{0,I'}$ is empty for any $I' \not\subseteq I = \{1, \dots, s+1\}$. Since \mathcal{X} is \mathbb{Q} -factorial, so is \mathcal{Y}_σ , and \mathcal{Y}_σ has only orbifold singularities, which is equivalent to the rational cone σ being simplicial (cf. [10]).

If $v_0, \dots, v_s \in N$ are primitive vectors belonging to 1-dimensional faces and generating σ in $N_\mathbb{R}$, we denote $N'_\sigma = \text{Span}_\mathbb{Z}\{v_0, \dots, v_s\}$ which is a sublattice of $N_\sigma = \mathbb{Z} \cdot (\sigma \cap N)$, and $M(\sigma) = \sigma^\perp \cap M \cong \mathbb{Z}^{n-s}$. Then $M \cong M_\sigma \oplus M(\sigma)$ where $M_\sigma = \text{hom}_\mathbb{Z}(N'_\sigma, \mathbb{Z}) \cong M/M(\sigma)$, and M_σ is a sublattice of $M'_\sigma = \text{hom}_\mathbb{Z}(N'_\sigma, \mathbb{Z}) = \text{Span}_\mathbb{Z}(v_0^*, \dots, v_s^*)$, where v_j^* is the dual vector of v_j . Note that the restriction of π on U is given by a monomial \mathcal{Z}^{u_σ} , where $u_\sigma \in \check{\sigma} \cap M_\sigma$ satisfies $\langle u_\sigma, v_j \rangle = 1$ for $j = 0, \dots, s$, i.e. $u_\sigma = \sum_{j=0}^s v_j^*$.

If $G = N_\sigma/N'_\sigma$, and $\mathcal{Y}'_\sigma = \text{Spec}(\mathbb{C}[\check{\sigma} \cap M'_\sigma]) \cong \mathbb{C}^{s+1}$, then the finite group G acts on \mathcal{Y}'_σ by $v \cdot \mathcal{Z}^u = \exp(2\pi\sqrt{-1}\langle v, u \rangle) \cdot \mathcal{Z}^u$ for any $v \in N_\sigma$ and $u \in M'_\sigma$, and $\mathcal{Y}'_\sigma/G \times (\mathbb{C}^*)^{n-s} \cong \mathcal{Y}_\sigma$. We denote $q_\sigma : \mathcal{Y}'_\sigma \times (\mathbb{C}^*)^{n-s} \rightarrow \mathcal{Y}_\sigma$ the quotient map of the G -action. Let $z_j = \mathcal{Z}^{v_j^*}$, $j = 0, \dots, s$, be coordinates on \mathcal{Y}'_σ , and z_{s+1}, \dots, z_n be coordinates on $(\mathbb{C}^*)^{n-s}$. The restriction $q_\sigma : T_{N'_\sigma} \times (\mathbb{C}^*)^{n-s} \rightarrow T_N$ is a finite covering map, where $T_N = N \otimes_\mathbb{Z} \mathbb{C}^*$ and $T_{N'_\sigma} = N'_\sigma \otimes_\mathbb{Z} \mathbb{C}^*$.

If we denote $Y_{\sigma,t} = \text{div}(\mathcal{Z}^{u_\sigma} - t)$, $t \in \mathbb{C}$, then $Y_{\sigma,t} \cap U = X_t \cap U$, and $Y_{\sigma,0} = \sum_{j=0}^s D_j$ where D_j is a primitive toric Weil divisor of Y_σ . The restriction $q_\sigma : q_\sigma^{-1}(Y_{\sigma,t}) \rightarrow Y_{\sigma,t}$ is a finite covering map as $X_t \cap U \subset T_N$ when $t \neq 0$, and $q_\sigma^{-1}(Y_{\sigma,t})$ is given by the equation $z_0 \cdots z_s = t$ in $\mathcal{Y}'_\sigma \times (\mathbb{C}^*)^{n-s}$. We can regard z_1, \dots, z_n as coordinates of $q_\sigma^{-1}(Y_{\sigma,t})$ for any $t \neq 0$. We assume that

$U \subset \mathcal{Y}_\sigma$ satisfies $q_\sigma^{-1}(U) = \{(z_0, \dots, z_s) \in \mathcal{Y}'_\sigma \mid |z_j| < \epsilon, 0 \leq j \leq s\} \times (U \cap X_{0,I})$ for an $\epsilon < 1$ without loss of generality.

Let x_0, \dots, x_s be coordinates on $N'_{\sigma, \mathbb{R}} = N'_\sigma \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{s+1}$ respecting to the basis v_0, \dots, v_s . Note that the interior of the s -dimensional cell $\overline{B}_I \in \mathcal{B}$ associated to $X_{0,I}$ is given by

$$\begin{aligned}
 B_I &= \{v \in \text{int}(\sigma) \mid \langle v, u_\sigma \rangle = 1\} \\
 &= \{(x_0, \dots, x_s) \in \mathbb{R}^{s+1} \mid \sum_{j=0}^s x_j = 1, x_j > 0, j = 0, \dots, s\} \\
 (3.7) \quad &= \{(x_1, \dots, x_s) \in \mathbb{R}^s \mid \sum_{j=1}^s x_j < 1, x_j > 0, j = 1, \dots, s\}.
 \end{aligned}$$

Here we regard x_1, \dots, x_s as coordinates on B_I .

For any $t \in \Delta^*$, we define the covering map

$$(3.8) \quad P_t : \mathbb{C}^s \times (\mathbb{C}^*)^{n-s} \rightarrow q_\sigma^{-1}(Y_{\sigma,t})$$

by letting $z_j = e^{(\log |t|)w_j}$ and $x_j = \text{Re}(w_j)$, $j = 1, \dots, s$, i.e.

$$P_t(w_1, \dots, w_s, z_{s+1}, \dots, z_n) = (e^{(\log |t|)w_1}, \dots, e^{(\log |t|)w_s}, z_{s+1}, \dots, z_n).$$

The fundamental domains of P_t are

$$(3.9) \quad \mathfrak{D}_{t,\nu} = \{(w_1, \dots, w_s) \in \mathbb{C}^s \mid \frac{2\pi\nu}{\log |t|} \leq \text{Im}(w_j) \leq \frac{2\pi(\nu+1)}{\log |t|}\} \times (\mathbb{C}^*)^{n-s}$$

for $\nu \in \mathbb{Z}$, and naturally $(\mathbb{C}^s / \sqrt{-1} \frac{2\pi\mathbb{Z}^s}{\log |t|}) \times (\mathbb{C}^*)^{n-s}$ is biholomorphic to $q_\sigma^{-1}(Y_{\sigma,t})$ by further setting $z_0 = tz_1^{-1} \dots z_s^{-1} = t \exp(-\sum_{j=1}^s (\log |t|)w_j)$.

Note that if $|z_j| < \epsilon$, $j = 0, \dots, s$, then $x_j > \frac{\log \epsilon}{\log |t|}$ for $j = 1, \dots, s$, and $\log |t|(1 - \sum_{j=1}^s x_j) = \log |z_0| < \log \epsilon$, which implies

$$P_t^{-1}(q_\sigma^{-1}(Y_{\sigma,t} \cap U)) = B_t \times \sqrt{-1}\mathbb{R}^s \times (\mathbb{C}^*)^{n-s},$$

where

$$B_t = \{(x_1, \dots, x_s) \in \mathbb{R}^s \mid x_j > \frac{\log \epsilon}{\log |t|}, j = 1, \dots, s, 1 - \sum_{j=1}^s x_j > \frac{\log \epsilon}{\log |t|}\} \subset B_I.$$

Hence

$$q_\sigma^{-1}(Y_{\sigma,t} \cap U) \subset B_I \times \sqrt{-1}(\mathbb{R}^s / \frac{2\pi\mathbb{Z}^s}{\log |t|}) \times (\mathbb{C}^*)^{n-s} \subset q_\sigma^{-1}(Y_{\sigma,t}).$$

Lemma 3.2. *Let $K \subset B_I$ be a compact subset such that $K \subset B_t$ for $|t| \ll 1$. On $K \times \sqrt{-1}\mathbb{R}^s \times (U \cap X_{0,I}) \subset (q_\sigma \circ P_t)^{-1}(U \cap Y_{\sigma,t})$, when $t \rightarrow 0$,*

i)

$$P_t^* q_\sigma^* \chi_t V_t \rightarrow V'_0 = \frac{1}{4(1 - \sum_{j=1}^s x_j)^2} \prod_{j=1}^s \frac{dw_j \wedge d\bar{w}_j}{4x_j^2} \wedge V_I,$$

in the C^∞ -sense, where V_I is a smooth volume form on $U \cap X_{0,I}$.

ii)

$$P_t^* q_\sigma^* \tilde{\omega}_t \rightarrow \omega_{U,I}^o + \frac{\sqrt{-1}}{2} \left(\sum_{j=1}^s \frac{dw_j \wedge d\bar{w}_j}{x_j^2} + \frac{\sum_{i,j=1}^s dw_i \wedge d\bar{w}_j}{(1 - \sum_{j=1}^s x_j)^2} \right) = \tilde{\omega}^o$$

in the C^∞ -sense, where $\omega_{U,I}^o$ is the pull-back of the complete Kähler metric $\tilde{\omega}_{0,I}$ on $U \cap X_{0,I}$.

Proof. Let $w_0 = 1 + \sqrt{-1} \frac{\arg(t)}{\log|t|} - w_1 - \dots - w_s$ on $(q_\sigma \circ P_t)^{-1}(Y_{\sigma,t})$, and $x_0 = 1 - x_1 - \dots - x_s$ on B_I . We have $z_0 = e^{\log|t|w_0}$ and $dw_0 = -dw_1 - \dots - dw_s$ on $(q_\sigma \circ P_t)^{-1}(Y_{\sigma,t})$.

Now, we claim that for a smooth function λ on $\mathcal{Y}'_\sigma \times (\mathbb{C}^*)^{n-s}$, $\lambda \circ P_t \rightarrow \lambda' = \lambda(0, z_{s+1}, \dots, z_n)$ and $dz_j = \frac{\partial z_j}{\partial w_j} dw_j \rightarrow 0$, $j = 0, \dots, s$, in the C^∞ -sense on any compact subset of $(q_\sigma \circ P_t)^{-1}(U \cap Y_{\sigma,t})$, when $t \rightarrow 0$. Since

$$\left| \frac{\partial^k z_j}{\partial w_j^k} \right| = |(\log|t|)^k e^{(\log|t|)x_j}| \leq |(\log|t|)^k| e^{\varepsilon_j \log|t|} \rightarrow 0, \quad \left| \frac{\partial z_0}{\partial w_j} \right| = \left| \frac{\partial z_0}{\partial w_0} \right|$$

for a $\varepsilon_j > 0$, $0 \leq j \leq s$, the claim follows by $\left| \frac{\partial^k \lambda}{\partial z_{i_1}^{k_1} \dots \partial z_{i_{s'}}^{k_{s'}}} \right| \leq C$ for some

constants $C > 0$.

Since \mathcal{Y}_σ has only Gorenstein orbifold singularities, for the generator $\Omega_\sigma \in \mathcal{O}(\mathcal{K}_{\mathcal{Y}_\sigma})$, $q_\sigma^* \Omega_\sigma$ is a G -invariant no-where vanishing holomorphic $(n+1, 0)$ -form on $\mathcal{Y}'_\sigma \times (\mathbb{C}^*)^{n-s}$, and thus

$$q_\sigma^* V = \eta \prod_{j=0}^n dz_j \wedge d\bar{z}_j,$$

where $\eta > 0$ is a smooth function on $q_\sigma^{-1}(U)$. We obtain

$$q_\sigma^* V_t = \eta \prod_{j=1}^s \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2} \wedge \prod_{i=s+1}^n dz_i \wedge d\bar{z}_i$$

on $q_\sigma^{-1}(X_t \cap U)$.

Without loss of generality, we assume that $I = \{1, \dots, s+1\}$. Under a local trivialization of $\mathcal{O}(\mu X_{0,i})$, $i \in I$, on U , we have that $q_\sigma^* s_i = z_j^\mu$, where $j = i - 1$, and the Hermitian metric $\|\cdot\|_i$ is given by restricting a smooth function ρ'_j on an open subset \mathbb{C}^ν for a local embedding $U \hookrightarrow \mathbb{C}^\nu$. Thus $q_\sigma^* \alpha_{j+1} = \log \rho_j |z_j|^2$ for $j = 0, \dots, s$, where $\rho_j = \rho'_j \circ q_\sigma > 0$ are smooth

function on $q_\sigma^{-1}(U)$, and $q_\sigma^* \alpha_i < 0$, $i = s+2, \dots, l$, are also smooth functions. By (3.3),

$$q_\sigma^* \chi_t V_t = \eta'' \frac{(\log |t|^2)^2}{(\log(\rho_0 |z_0|^2))^2} \prod_{j=1}^s \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2 (\log(\rho_j |z_j|^2))^2} \wedge \prod_{i=s+1}^n dz_i \wedge d\bar{z}_i,$$

where $\eta'' > 0$ is a smooth function on $q_\sigma^{-1}(U)$, and

$$P_t^* q_\sigma^* \chi_t V_t = \frac{\eta'' \circ P_t}{(\frac{\log \rho_0}{\log |t|^2} + 2x_0)^2} \prod_{j=1}^s \frac{dw_j \wedge d\bar{w}_j}{(\frac{\log \rho_j}{\log |t|^2} + 2x_j)^2} \wedge \prod_{i=s+1}^n dz_i \wedge d\bar{z}_i.$$

By taking $t \rightarrow 0$, we obtain the convergence of volume forms.

We have $q_\sigma^* \omega^o$ is a smooth $(1, 1)$ -form on $q_\sigma^{-1}(U)$, and

$$P_t^* q_\sigma^* \omega_t^o = \sqrt{-1} P_t^* q_\sigma^* \partial \bar{\partial} \log V_t = \sqrt{-1} \partial \bar{\partial} \log \eta' \rightarrow \sqrt{-1} \partial \bar{\partial} \log \eta',$$

in the C^∞ -sense, when $t \rightarrow 0$, where $\eta' = \eta(0, z_{s+1}, \dots, z_n) > 0$. Note that $\sqrt{-1} \partial \bar{\partial} \log \eta'$ is the pull-back of $\omega^o|_{X_{0,I} \cap U}$. Since $q_\sigma^* \frac{\text{Ric}(\|\cdot\|_i)}{\alpha_i}$ and $q_\sigma^* \frac{\partial \alpha_i \wedge \bar{\partial} \alpha_i}{\alpha_i^2}$, $i = s+2, \dots, l$, are also smooth $(1, 1)$ -forms on $q_\sigma^{-1}(U)$, we have

$$P_t^* q_\sigma^* \left(\frac{\text{Ric}(\|\cdot\|_i)}{\alpha_i} + \sqrt{-1} \frac{\partial \alpha_i \wedge \bar{\partial} \alpha_i}{\alpha_i^2} \right) \rightarrow \beta_i,$$

in the C^∞ -sense, where β_i is the pull-back of the smooth Kähler form $(\frac{\text{Ric}(\|\cdot\|_i)}{\alpha_i} + \sqrt{-1} \frac{\partial \alpha_i \wedge \bar{\partial} \alpha_i}{\alpha_i^2})|_{U \cap X_{0,I}}$ on $U \cap X_{0,I}$. Thus

$$\omega_{U,I}^o = \sqrt{-1} \partial \bar{\partial} \log \eta' + 2 \sum_{i=s+2}^l \beta_i$$

is the pull-back of the restriction of $\tilde{\omega}_{0,I}$ on $U \cap X_{0,I}$ by (3.5).

On K , $(\log |t|)x_j \rightarrow -\infty$, $j = 0, \dots, s$, and thus,

$$P_t^* q_\sigma^* \frac{\text{Ric}(\|\cdot\|_{j+1})}{\alpha_{j+1}} = \frac{\sqrt{-1} \partial \bar{\partial} \log \rho_i}{\log \rho_i + 2(\log |t|)x_i} \rightarrow 0,$$

in the C^∞ -sense. Furthermore,

$$P_t^* q_\sigma^* \frac{\partial \alpha_{j+1} \wedge \bar{\partial} \alpha_{j+1}}{\alpha_{j+1}^2} = \frac{(\partial \log \rho_j + \log |t| dw_j) \wedge (\bar{\partial} \log \rho_j + \log |t| d\bar{w}_j)}{(\log \rho_j + 2(\log |t|)x_j)^2} \rightarrow \frac{dw_j \wedge d\bar{w}_j}{4x_j^2},$$

in the C^∞ -sense, when $t \rightarrow 0$. Thus we obtain the conclusion by (3.4), and

$$\frac{dw_0 \wedge d\bar{w}_0}{4x_0^2} = \frac{\sum_{i,j=1}^s dw_i \wedge d\bar{w}_j}{4(1 - \sum_{j=1}^s x_j)^2}.$$

□

Lemma 3.3. *Let φ_t be the unique solution of (3.6), and $\omega_t = \tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$. For any sequence $t_k \rightarrow 0$, a subsequence of $\varphi_{t_k} \circ q_\sigma \circ P_{t_k}$ converges to φ_0 in the C^∞ -sense on $K \times \sqrt{-1}\mathbb{R}^s \times (U \cap X_{0,I})$, where φ_0 is a smooth function on $B_I \times \sqrt{-1}\mathbb{R}^s \times (U \cap X_{0,I})$ satisfying the complex Monge-Ampère equation*

$$(3.10) \quad (\tilde{\omega}^o + \sqrt{-1}\partial\bar{\partial}\varphi_0)^n = e^{\varphi_0} V_0',$$

$$\text{with } |\varphi_0| \leq C_3, \text{ and } C_4^{-1}\tilde{\omega}^o \leq \tilde{\omega}^o + \sqrt{-1}\partial\bar{\partial}\varphi_0 \leq C_4\tilde{\omega}^o.$$

Furthermore, φ_0 is independent of $\text{Im}(w_j)$, $j = 1, \dots, s$, i.e.

$$\varphi_0 = \varphi_0(x_1, \dots, x_s, z_{s+1}, \dots, z_n).$$

Proof. By Proposition 3.1, we have that

$$|\varphi_t| \leq C, \text{ and } C^{-1}P_t^*q_\sigma^*\tilde{\omega}_t \leq P_t^*q_\sigma^*(\tilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t) \leq CP_t^*q_\sigma^*\tilde{\omega}_t$$

for a constant $C > 0$. We obtain the $C^{2,\alpha}$ -estimates for φ_t , i.e. $\|\varphi_t \circ q_\sigma \circ P_t\|_{C^{2,\alpha}} \leq \bar{C}$, by Lemma 3.2 and the Evans-Krylov theory (cf. [11, 33]), and the higher order estimates $\|\varphi_t \circ q_\sigma \circ P_t\|_{C^\nu} \leq C(\nu)$ by the standard Schauder estimates on any compact subset $K' \subset K \times \sqrt{-1}\mathbb{R}^s \times (U \cap X_{0,I})$. Thus by passing to a subsequence of t_k , $\varphi_{t_k} \circ q_\sigma \circ P_{t_k}$ converges to a smooth function φ_0 in the locally C^∞ -sense, and φ_0 satisfies the the complex Monge-Ampère equation (3.10) by Lemma 3.2.

Since $\varphi_t \circ q_\sigma \circ P_t$ is a periodic function with period $\sqrt{-1}\frac{2\pi\mathbb{Z}^s}{\log|t|}$, i.e.

$$\varphi_t \circ q_\sigma \circ P_t(w, z) = \varphi_t \circ q_\sigma \circ P_t(w + \sqrt{-1}\frac{2\pi\mathbf{m}}{\log|t|}, z),$$

for any $\mathbf{m} \in \mathbb{Z}^s$, where $w = (w_1, \dots, w_s)$ and $z = (z_{s+1}, \dots, z_n)$, we obtain that φ_0 is independent of $\text{Im}(w_j)$, $j = 1, \dots, s$, by the smooth convergence. \square

Since $\frac{\partial^2\varphi_0}{\partial w_i \partial w_j} = \frac{\partial^2\varphi_0}{4\partial x_i \partial x_j}$, the corresponding Riemannian metric of $\tilde{\omega}^o + \sqrt{-1}\partial\bar{\partial}\varphi_0$ is

$$(3.11) \quad g_0 = \sum_{i,j=1}^s \left(\frac{\delta_{ij}}{x_i^2} + \frac{1}{(1 - \sum_{j=1}^s x_j)^2} + \frac{\partial^2\varphi_0}{2\partial x_i \partial x_j} \right) (dx_i dx_j + d\theta_i d\theta_j) + \mathcal{G}_0,$$

where $\theta_j = \text{Im}(w_j)$, $j = 1, \dots, n$, and \mathcal{G}_0 denotes the remaining terms that do not involve any $d\theta_i d\theta_j$ and $dx_i dx_j$.

Note that both $\tilde{\omega}^o$ and φ_0 are invariant under the translation $w_j \mapsto w_j + \lambda_j \sqrt{-1}$, $j = 1, \dots, s$, for any $(\lambda_1, \dots, \lambda_s) \in \mathbb{R}^s$. Hence for any $t \neq 0$, $\tilde{\omega}^o + \sqrt{-1}\partial\bar{\partial}\varphi_0$ descends to a Kähler metric ω_t^{sf} on $Y_{\sigma,t} \cap U$, which satisfies that

$$(3.12) \quad P_t^*q_\sigma^*\omega_t^{sf} = \tilde{\omega}^o + \sqrt{-1}\partial\bar{\partial}\varphi_0, \text{ and } \|\omega_{t_k} - \omega_{t_k}^{sf}\|_{C_{loc}^\nu(Y_{\sigma,t_k} \cap U, \omega_{t_k}^{sf})} \rightarrow 0,$$

for any $\nu > 0$, when $t_k \rightarrow 0$ by Lemma 3.3.

Define a fibration

$$\tilde{f}_t : B_I \times \sqrt{-1}(\mathbb{R}^s / (\frac{2\pi\mathbb{Z}^s}{\log|t|})) \times (\mathbb{C}^*)^{n-s} \rightarrow B_I \times (\mathbb{C}^*)^{n-s}$$

by the projection. Note that \tilde{f}_t is G -equivariant, \tilde{f}_t induces a T^s -fibration

$$(3.13) \quad f_t : U \cap Y_{\sigma,t} \rightarrow B_t \times (\mathbb{C}^*)^{n-s}, \quad \text{with } \tilde{f}_t = f_t \circ q_\sigma.$$

For a point $(x, z) \in B_t \times (\mathbb{C}^*)^{n-s}$, where $x = (x_1, \dots, x_s) \in B_t$ and $z = (z_{s+1}, \dots, z_n) \in (\mathbb{C}^*)^{n-s}$, the fiber $f_t^{-1}(x, z)$ satisfies that $(q_\sigma \circ P_t)^{-1}(f_t^{-1}(x, z)) = \{(x + \sqrt{-1}\theta, z) | \theta = (\theta_1, \dots, \theta_s) \in \mathbb{R}^s\}$. Hence the restriction of the Kähler metric ω_t^{sf} on $f_t^{-1}(x, z)$ is a flat Riemannian metric, i.e. ω_t^{sf} is a semi-flat metric, and

$$\text{diam}_{\omega_{t_k}}(f_{t_k}^{-1}(x, z)) \sim \text{diam}_{\omega_{t_k}^{sf}}(f_{t_k}^{-1}(x, z)) \leq \frac{2\pi s \sqrt{C_{x,z}}}{-\log|t|} \rightarrow 0,$$

when $t \rightarrow 0$, by (3.12) and (3.11), where

$$C_{x,z} = \sum_{ij=1}^s \left| \frac{\delta_{ij}}{x_i^2} + \frac{1}{(1 - \sum_{j=1}^s x_j)^2} + \frac{\partial^2 \varphi_0}{2\partial x_i \partial x_j}(x, z) \right|.$$

We denote $W_U = B_I \times (U \cap X_{0,I})$, and naturally regard $W_U \subset B_I \times \sqrt{-1}(\mathbb{R}^s / (\frac{2\pi\mathbb{Z}^s}{\log|t|})) \times (U \cap X_{0,I})$ given by $\theta_j = 0$, $j = 1, \dots, n$. We let $g_{W_U} = g_0|_{W_U}$. If $p \in W_U$, and $r > 0$ such that the metric ball $B_{g_{W_U}}(p, r) \subset K''$ for a compact subset $K'' \subset W_U$, then

$$(3.14) \quad (B_{\omega_{t_k}}(p_{t_k}, r), \omega_{t_k}) \quad \text{and} \quad (B_{\omega_{t_k}^{sf}}(p_{t_k}, r), \omega_{t_k}^{sf}) \rightarrow (B_{g_{W_U}}(p, r), g_{W_U})$$

in the Gromov-Hausdorff sense by (3.12), when $t_k \rightarrow 0$, for some $p_t \in X_t \cap U$. By Gromov's precompactness theorem (cf. [12]), $(X_{t_k}, \omega_{t_k}, p_{t_k})$ converges to a complete metric space (W, d_W, p_∞) of Hausdorff dimension ϱ in the pointed Gromov-Hausdorff sense, and there is a local isometric embedding $(B_{g_{W_U}}(p_\infty, r), g_{W_U}) \hookrightarrow (W, d_W)$, which implies $\varrho = \dim_{\mathbb{R}} B_I \times X_{0,I}$.

In summary, we have the following proposition.

Proposition 3.4. *There is a semi-flat Kähler-Einstein metric ω_t^{sf} on $X_t \cap U$ respecting to f_t such that*

$$\|\omega_{t_k} - \omega_{t_k}^{sf}\|_{C_{loc}^\nu(Y_{\sigma,t_k} \cap U, \omega_{t_k}^{sf})} \rightarrow 0,$$

for any $\nu > 0$, and a sequence $t_k \rightarrow 0$. Furthermore, $(X_{t_k}, \omega_{t_k}, p_{t_k})$ converges to a complete metric space (W, d_W, p_∞) in the pointed Gromov-Hausdorff sense by choosing some base points $p_t \in X_t$, and the Hausdorff dimension of W equals to $\dim_{\mathbb{R}} B_I + 2 \dim_{\mathbb{C}} X_{0,I}$.

Now we are ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\{U_\gamma\}$ be an open cover of $X_{0,I}^o$ such that any $U_\gamma \subset \mathcal{X}$ is isomorphic to an open subset of a toric variety, and does not intersect with $\bigcup_{i \notin I} X_{0,i}$. By applying the above arguments to U_γ , we have

$W_{U_\gamma} = B_I \times (U_\gamma \cap X_{0,I}) \subset B_I \times \sqrt{-1}(\mathbb{R}^s / (\frac{2\pi\mathbb{Z}^s}{\log|t|})) \times (U_\gamma \cap X_{0,I})$, and a metric $g_{U_\gamma} = g_{\gamma,0}|_{W_{U_\gamma}}$, where $g_{\gamma,0}$ is the Riemannian metric given by (3.11).

If $\omega_{\gamma,t}^{sf}|_{X_t \cap U_\gamma}$ denotes the semi-flat Kähler-Einstein metric satisfying (3.12), then, by Lemma 3.3, $P_t^* q_\sigma^* \omega_{\gamma,t}^{sf}$ is uniformly equivalent to $\tilde{\omega}^o$ on $B_I \times \sqrt{-1}\mathbb{R}^s \times (U_\gamma \cap X_{0,I})$. For any U_γ , since there are finite $U_1, \dots, U_\varsigma \in \{U_\gamma\}$ such that $(U \cup U_1 \cup \dots \cup U_\varsigma \cup U_\gamma) \cap X_t$ is connected, we have a point $p_{t_k, \gamma} \in X_{t_k} \cap U_\gamma$ such that $\text{dist}_{\omega_{t_k}}(p_{t_k, \gamma}, p_{t_k}) \leq C_\gamma$ for a constant independent of t_k by Proposition 3.4. Thus there is a local isometric embedding $\iota_\gamma : (W_{U_\gamma}, g_{U_\gamma}) \hookrightarrow (W, d_W)$. Note that the restriction of g_{U_γ} on any $B_I \times \{q\}$ is complete, $g_{U_\gamma}|_{U_\gamma \cap X_{0,I}}$ is uniformly equivalent to $\tilde{\omega}^o|_{U_\gamma \cap X_{0,I}} = \tilde{\omega}_{0,I}|_{U_\gamma \cap X_{0,I}}$, and $\tilde{\omega}_{0,I}$ is complete on $X_{0,I}^o$ by (3.5). Therefore, $\bigcup_\gamma \iota_\gamma(W_{U_\gamma}, g_{U_\gamma}) \subset (W, d_W)$ is a complete Riemannian manifold, which implies that $\bigcup_\gamma \iota_\gamma(W_{U_\gamma}, g_{U_\gamma}) = (W, d_W)$.

Now we assume $\dim_{\mathbb{C}} X_{0,I} = 0$, i.e. $s = n$. Then $W_U = B_I$, $\varphi_0 = \varphi_0(x_1, \dots, x_n)$ is a function on B_I , and we denote $g_{B_I} = g_{W_U}$. We need the following lemma to finish the proof.

Lemma 3.5. *If*

$$\phi = \frac{\varphi_0}{2} - \sum_{j=1}^n \log x_j - \log(1 - \sum_{j=1}^n x_j),$$

then

$$g_{B_I} = \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j$$

on B_I , and ϕ is the unique solution of the real Monge-Ampère equation

$$\det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = \kappa e^{2\phi}, \quad \phi|_{\partial \bar{B}_I} = +\infty,$$

for a constant $\kappa > 0$, i.e. ϕ is obtained in Theorem 2.1.

Proof. Note that $\frac{\partial \varphi_0}{\partial w_j} = \frac{\partial \varphi_0}{2 \partial x_j}$, and

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \varphi_0}{2 \partial x_i \partial x_j} + \frac{\delta_{ij}}{x_j^2} + \frac{1}{(1 - \sum_{j=1}^n x_j)^2}.$$

By (3.11), $g_{B_I, ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. By Lemma 3.2 and Lemma 3.3, we have

$$\det\left(\frac{\delta_{ij}}{x_j^2} + \frac{1}{(1 - \sum_{j=1}^n x_j)^2} + \frac{\partial^2 \varphi_0}{2 \partial x_i \partial x_j}\right) = \frac{e^{\varphi_0} 2^n \eta'}{4^{n+1} (1 - \sum_{j=1}^n x_j)^2 \prod_{j=1}^n x_j^2},$$

where $\eta' > 0$ is a constant.

Now Proposition 5.5 in [6] implies that φ_0 is the unique solution of (3.10), which implies the uniqueness of ϕ . \square

Note that g_{B_I} is a complete metric on B_I , and thus (W, d_W) is isometric to (B_I, g_{B_I}) . By the uniqueness of g_{B_I} , we have the convergence of Proposition 3.4 without passing to any sequence t_k . We obtain the conclusion. \square

4. PROOF OF THEOREM 2.6

Proof of Theorem 2.6. Since $\mathcal{K}_{\mathcal{X}/\Delta}$ is ample, there is a section Ω_t of $\mathcal{K}_{\mathcal{X}/\Delta}$ such that $D \cap X_{0,I} = \emptyset$ where $D = \text{div}(\Omega_t)$. Let $U \subset \mathcal{X}$ be a neighborhood of $X_{0,I}$ isomorphic to an open subset of a toric variety \mathcal{Y}_σ , denoted still by U , such that $U \cap X_{0,I'}$ is empty for any $I' \not\subseteq I = \{1, \dots, s+1\}$. We assume that $D \cap U = \emptyset$ by shrinking U if necessary.

We adopt the construction in Section 3.2. There is a toric variety $\mathcal{Y}'_\sigma \cong \mathbb{C}^{n+1}$ with coordinates z_0, \dots, z_n , and a finite group $G = N/N'$ acting on \mathcal{Y}'_σ . Let $q_\sigma : \mathcal{Y}'_\sigma \rightarrow \mathcal{Y}_\sigma$ be the finite quotient by $G = N/N'$, and $Y_{\sigma,t} \subset \mathcal{Y}_\sigma$ such that $Y_{\sigma,t} \cap U = X_t \cap U$ as in Section 3.2. Recall that $q_\sigma^{-1}(Y_{\sigma,t})$ is given by $z_0 \cdots z_n = t$, and $q_\sigma^{-1}(Y_{\sigma,t} \cap U) \subset B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log|t|}) \subset q_\sigma^{-1}(Y_{\sigma,t})$, where B_I is given by (3.13).

For a $p = (p_1, \dots, p_n) \in B_I$, we define an embedding

$$\mathfrak{i}_t : B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log|t|}) \hookrightarrow \mathbb{C}^n / (2\pi\sqrt{-1}\mathbb{Z}^n) = Y_\infty$$

by letting $\tilde{w}_j = (\log|t|)(w_j - p_j)$, $j = 1, \dots, n$. We identify $B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log|t|})$ with the image $\mathfrak{i}_t(B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log|t|})) \subset Y_\infty$ by \mathfrak{i}_t without any confusion.

Assume that $\tilde{\lambda}_t = \tilde{\lambda}_t(w_1, \dots, w_n)$ is a family of functions convergence smoothly to $\tilde{\lambda}_0 = \tilde{\lambda}_0(x_1, \dots, x_n)$ under the coordinates w_1, \dots, w_n when $t \rightarrow 0$, i.e. $\frac{\partial^k \tilde{\lambda}_t}{\partial w_{j_1}^{k_1} \cdots \partial w_{j_m}^{k_m}} \rightarrow \frac{\partial^k \tilde{\lambda}_0}{\partial w_{j_1}^{k_1} \cdots \partial w_{j_m}^{k_m}} = \frac{\partial^k \tilde{\lambda}_0}{2^k \partial x_{j_1}^{k_1} \cdots \partial x_{j_m}^{k_m}}$. Since $w_j = p_j + (\log|t|)^{-1} \tilde{w}_j$ and $\frac{\partial \tilde{\lambda}_t}{\partial \tilde{w}_j} = (\log|t|)^{-1} \frac{\partial \tilde{\lambda}_t}{\partial w_j}$, we have $\tilde{\lambda}_t \rightarrow \tilde{\lambda}_0(p_1, \dots, p_n)$ in the C^∞ -sense on any on any compact subset $K' \subset \mathfrak{i}_t(B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log|t|})) \subset Y_\infty$, when $t \rightarrow 0$.

Since \mathcal{Y}_σ has only Gorenstein orbifold singularities, for the local generator $\Omega_\sigma \in \mathcal{O}(\mathcal{K}_{\mathcal{Y}_\sigma})$, $q_\sigma^* \Omega_\sigma$ is a G -invariant no-where vanishing holomorphic $(n+1, 0)$ -form on \mathcal{Y}'_σ , and thus

$$q_\sigma^* \Omega_t = \Omega_\sigma \otimes (dt)^{-1} = \zeta \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n},$$

on $q_\sigma^{-1}(X_t \cap U)$, where $\zeta > 0$ is a holomorphic function on \mathcal{Y}'_σ . Note that $\zeta(w_1, \dots, w_n) \rightarrow \zeta(0)$ in the C^∞ -sense by the argument in the proof of Lemma 3.2. Thus

$$q_\sigma^* \Omega_t = \zeta d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_n \rightarrow \Omega_\infty = \zeta(0) d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_n,$$

in the C^∞ -sense, when $t \rightarrow 0$.

If we denote $L_0 = \{0\} \times \sqrt{-1}(\mathbb{R}^n / (2\pi\mathbb{Z}^n))$, then for any $|t| \ll 1$, there is a $\vartheta_t \in \mathbb{R}$ such that $e^{\sqrt{-1}\vartheta_t} \int_{L_0} q_\sigma^* \Omega_t \in \mathbb{R}$, which implies that $\int_{L_0} \text{Im}(e^{\sqrt{-1}\vartheta_0} \Omega_\infty) = 0$, $\int_{L_0} \text{Im}(e^{\sqrt{-1}\vartheta_t} q_\sigma^* \Omega_t) = 0$, and $[\text{Im}(e^{\sqrt{-1}\vartheta_t} q_\sigma^* \Omega_t)|_{L_0}] = 0$ in $H^n(L_0, \mathbb{R})$. Since $e^{\sqrt{-1}\vartheta_0} \zeta(0)$ is a constant, we have

$$\text{Im}(e^{\sqrt{-1}\vartheta_0} \Omega_\infty)|_{L_0} = \text{Im}(e^{\sqrt{-1}\vartheta_0} \zeta(0) d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_n)|_{L_0} \equiv 0.$$

By Lemma 3.2 and Lemma 3.3,

$$(\log |t|)^2 q_\sigma^* \omega_t \rightarrow \frac{\sqrt{-1}}{2} \sum_{ij=1}^n \left(\frac{\delta_{ij}}{p_j^2} + \frac{1}{(1 - \sum_{j=1}^s p_j)^2} + \frac{\partial^2 \varphi_0}{2\partial x_i \partial x_j}(p) \right) d\tilde{w}_i \wedge d\tilde{w}_j = \omega_\infty,$$

in the C^∞ -sense on any compact subset K' on Y_∞ , when $t \rightarrow 0$. Since the curvature of ω_t are uniformly bounded independent of t , we have that ω_∞ is a flat metric on Y_∞ . A direct calculation shows $\omega_\infty|_{L_0} \equiv 0$. Note that for any $A \in H_2(L_0, \mathbb{Z})$,

$$|(\log |t|)^2 \int_A q_\sigma^* \omega_t| \rightarrow \left| \int_A \omega_\infty \right| = 0, \quad \text{and}$$

$$\int_A q_\sigma^* \omega_t = 2\pi \int_{q_\sigma(A)} c_1(\mathcal{K}_{X_t}) \in 2\pi\mathbb{Z}.$$

Thus $\int_A q_\sigma^* \omega_t = 0$, and $[q_\sigma^* \omega_t|_{L_0}] = 0$ in $H^2(L_0, \mathbb{R})$. By Theorem 10.8 in [14], we obtain a family of generalized special lagrangian submanifolds $\tilde{L}_t \subset B_I \times \sqrt{-1}(\mathbb{R}^n / \frac{2\pi\mathbb{Z}^n}{\log |t|})$ respecting to $q_\sigma^* \omega_t$ and $e^{\sqrt{-1}\vartheta_t} q_\sigma^* \Omega_t$ for $|t| \ll 1$. We obtain the conclusion by letting $L_t = q_\sigma(\tilde{L}_t)$. \square

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